Structure and Synthesis of Optimization Algorithms

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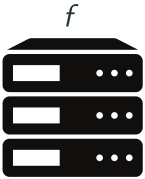
Optimization Setup

Want to solve an optimization problem $\beta^* \in \operatorname{argmin}_{\beta \in \mathbb{R}^c} f(\beta)$



Examples: Motion planning, regression, filtering

Challenge 1

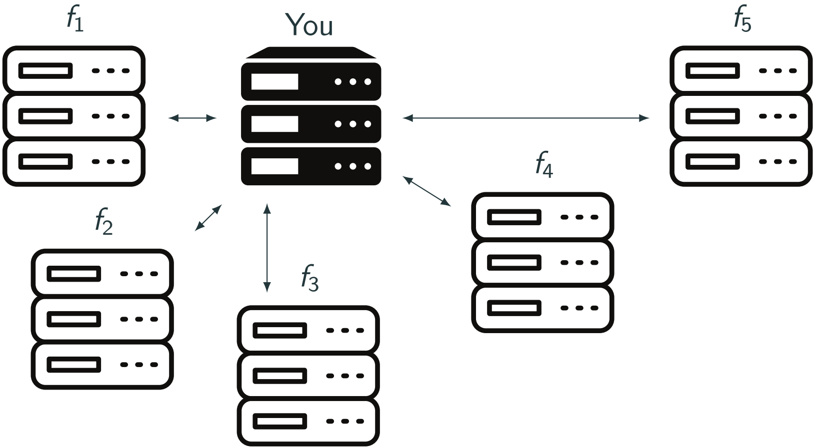
Want to solve an optimization problem $\beta^* \in \operatorname{argmin}_{\beta \in \mathbb{R}^c} f(\beta)$

Issue: f is only accessible over a **network**

Examples: teleoperation, satellite control, data assimilation

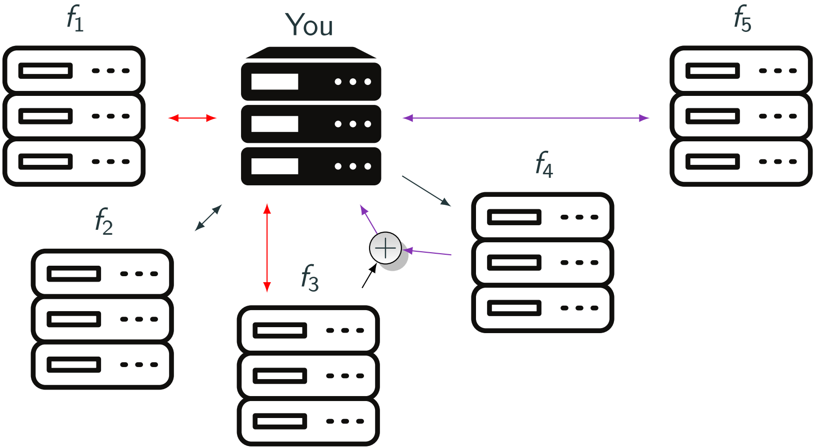
Challenge 2

Federated/composite $\beta^* \in \text{argmin}_{\beta \in \mathbb{R}^c} \sum_{i=1}^{s} f_i(\beta)$



Challenge 3, 4, 5, 6, ...

Memory, cross-talk, time delays, other nightmares



Main Ideas

Perform automated, disciplined design of fast algorithms

Restrict search using necessary convergence conditions

Accomplish by continuing a link of Optimization and Control

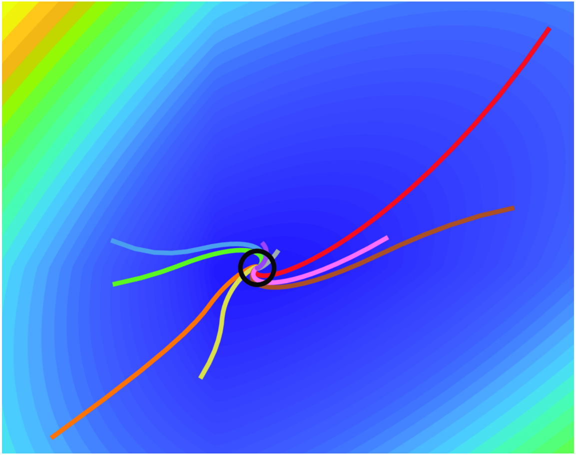
A Play in Five Acts

1. Convergence
2. Structure
3. Networks
4. Extensions
5. Analysis and Synthesis

Prelude: Algorithm Setting

Algorithm Definition (Simple)

Procedure to generate $\{\beta_k\}_{k\in\mathbb{N}}$, convergent if $\beta_k \to \beta^*$



Algorithmic Interconnection

Linear system interconnected[1] with map $F(z) = \sum_{i=1}^{s} \partial f_i(z^i)$:

$$\left(\begin{array}{c} x_{k+1} \\ \hline z_k \end{array}\right) = \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) \left(\begin{array}{c} x_k \\ \hline w_k \end{array}\right), \quad w_k \in F(z_k)$$

[1]Wang, Jing, and Nicola Elia. "A control perspective for centralized and distributed convex optimization." 2011 50th IEEE conference on decision and control and European control conference. IEEE, 2011.

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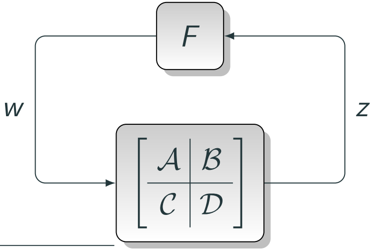

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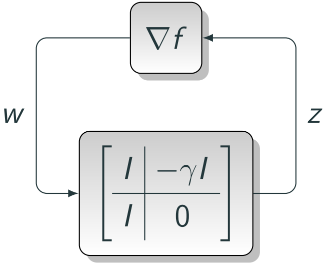
Interconnections Examples: GD/PPM

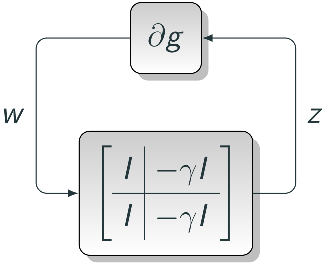
Gradient Descent

$x_{k+1} = x_k - \gamma \nabla f(x_k)$



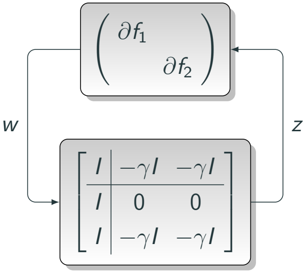
Proximal Point Method

$x_{k+1} = (I + \gamma \partial g)^{-1}(x_k)$

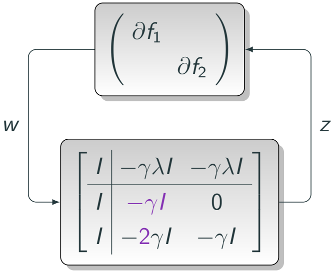


Interconnection Examples: PGD/Douglas-Rachford

Projected Gradient Descent



Douglas-Rachford[2]



[2]Douglas, J., & Rachford, H. H. (1956). On the numerical solution of heat conduction problems in two and three space variables. Transactions of the American mathematical Society, 82(2), 421-439.

Act 1: Convergence

Fixed Points and Convergence

Fixed point (x^*, w^*, z^*) of algorithmic interconnection:

$$\left(\begin{array}{c} x^* \\ \hline z^* \end{array}\right) = \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) \left(\begin{array}{c} x^* \\ \hline w^* \end{array}\right), \qquad w^* \in F(z^*)$$

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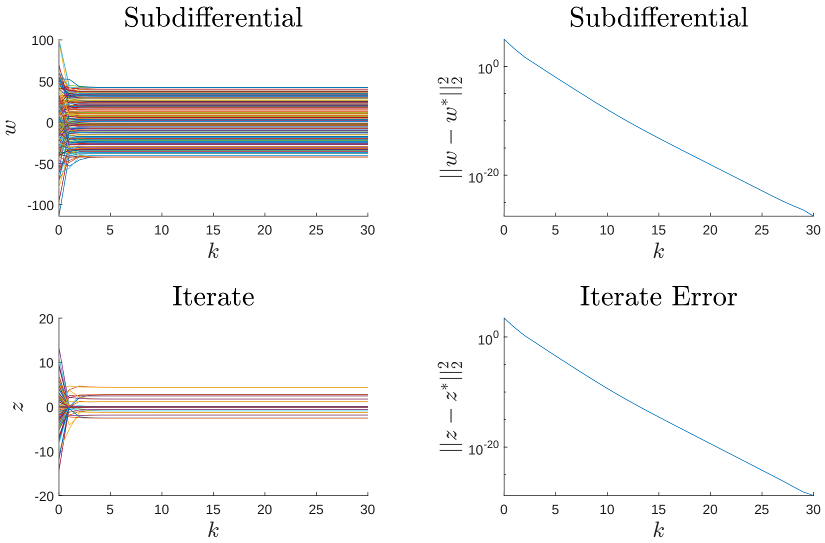
Convergent: (x^*, w^*, z^*) exists with[3]

$$\begin{aligned} \text{Consensus:} \quad & z^{1*} = z^{2*} = \ \dots \ = z^{s*} \\ \text{Optimality:} \quad & \textstyle\sum_{i=1}^{s} w^{i*} = 0 \\ \text{(Strong) Attractivity}^{*}\text{:} \quad & \forall x_0 : (x, w, z) \to (x^*, w^*, z^*) \end{aligned}$$

[3] Upadhyaya, Manu, et al. "Automated tight Lyapunov analysis for first-order methods." Mathematical Programming 209.1 (2025): 133-170.

Convergent Execution



PGD solving hard LASSO: $\min_{\|\beta\|_1 \le 20} \frac{1}{2}\|Q\beta - b\|_2^2$

Convergence Conditions

Incremental *class* for f (e.g. each f_i strongly convex/smooth)

Convergent algorithm for all f in the *class* must satisfy

1. **Robust Stability** Dynamics (difficult)
2. **Regulator Equation** Linear Algebra (easy)

Assumptions

1. $(\mathcal{A}, \mathcal{B})$ stabilizable, $(\mathcal{A}, \mathcal{C})$ detectable
2. (β^*, w^*) unique for each f in *class*
3. Well-posed: $(F^{-1} - \mathcal{D})^{-1}$ is continuous $\forall f$ in *class*
4. Known quadratic $\sum_{i=1}^{s} \frac{m_i}{2} \|\beta - \beta_i^*\|_2^2$ is in *class*

Unconstrained Setting

Optimization of a quadratic $f(\beta) = \frac{m}{2}\|\beta - \beta^*\|_2$ with $m > 0$

$$\left(\frac{x_{k+1}}{z_k}\right) = \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) \left(\frac{x_k}{w_k}\right), \quad w_k = m(z_k - \beta^*) \ldots$$

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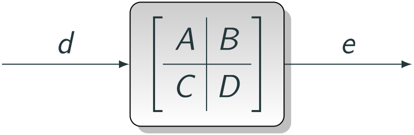
...occurs if *disturbance rejection* is achieved

$$\left(\begin{array}{c} x_{k+1} \\ \hline z_k - \beta^* \end{array}\right) = \left(\left(\begin{array}{c|c} \mathcal{A} & 0 \\ \hline \mathcal{C} & I \end{array}\right) + \mathrm{Stuff}(m) \left(\begin{array}{c|c} \mathcal{C} & I \end{array}\right)\right) \left(\begin{array}{c} x_k \\ \hline -\beta^* \end{array}\right).$$

Interlude: Disturbance Rejection

Review: Linear Regulation (Constant Disturbance)

Rejection of constant disturbances d (e.g. setpoint tracking)



Want to satisfy two properties:

1. **Stability:** $x \to 0$ when $d = 0$,
2. **Output Regulation:** Error $e \to 0$ even when $d \neq 0$.

Disturbance Rejection (Constant Disturbance)

Necc. and suff.[4] for disturbance rejection in linear systems:

1. **Stability:** A is Schur ($\rho(A) < 1$)
2. **Regulator Equation:** Exists unique matrix $\mathbf{X}$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ I \end{pmatrix} = \begin{pmatrix} \mathbf{X} \\ 0 \end{pmatrix}$$

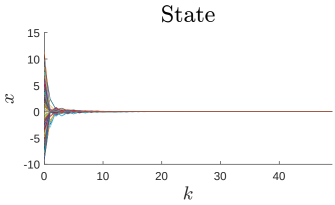
Asymptotic properties: $x \to \mathbf{X}d, \; e \to 0$ for all (x_0, d)

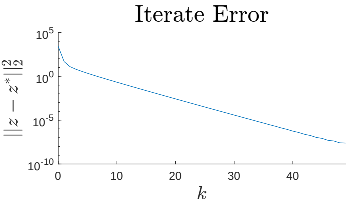
[4] Francis, Bruce A. "The linear multivariable regulator problem." SIAM Journal on Control and Optimization 15.3 (1977): 486-505. Includes non-constant persistent disturbances and the generalization to controller synthesis.

Back to the Show

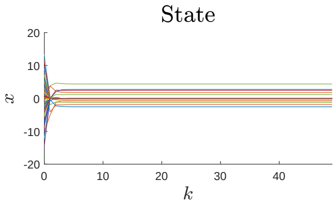
Convergence Analogues

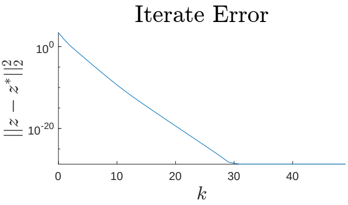
Stability: $x \to 0$ if $(\beta^*, w^*) = 0$





Output Regulation: $z - z^* \to 0$ always





Unconstrained Interpretation

Disturbance rejection with $d := -\beta^*$ implies convergence

$$\left(\begin{array}{c} x_{k+1} \\ \hline z_k - \beta^* \end{array}\right) = \left(\left(\begin{array}{c|c} \mathcal{A} & 0 \\ \hline \mathcal{C} & I \end{array}\right) + \mathrm{Stuff}(m)\left(\begin{array}{c|c} \mathcal{C} & I \end{array}\right)\right)\left(\begin{array}{c} x_k \\ \hline -\beta^* \end{array}\right),$$

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Regulator equation is independent of m

$$\exists \mathbf{X} : \qquad \begin{pmatrix} \mathcal{A} & 0 \\ \mathcal{C} & I \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ I \end{pmatrix} = \begin{pmatrix} \mathbf{X} \\ 0 \end{pmatrix}$$

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Generalizes to *classes* of functions f under well-posedness

Convergence: Composite Optimization

Robust Stability: $x \to 0$ for all x_0 if $(\beta^*, w^*) = 0$

$$\left(\begin{array}{c} x_{k+1} \\ \hline z_k \end{array}\right) = \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) \left(\begin{array}{c} x_k \\ \hline w_k \end{array}\right) \qquad w_k \in F(z_k)$$

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Regulator Equation [5]: with consensus N, $(\mathbf{1}_s \otimes I)^\top N = 0$

$$\exists \mathbf{X} : \left(\begin{array}{c:cc} \mathcal{A} & 0 & \mathcal{B}N \\ \mathcal{C} & \mathbf{1}_s \otimes I_c & \mathcal{D}N \end{array}\right) \begin{pmatrix} \mathbf{X} \\ \hdashline I \end{pmatrix} = \begin{pmatrix} \mathbf{X} \\ 0 \end{pmatrix},$$

Disturbance $d = (-\beta^*, w^{1*}, w^{2*}, \ldots, w^{(s-1)*})$

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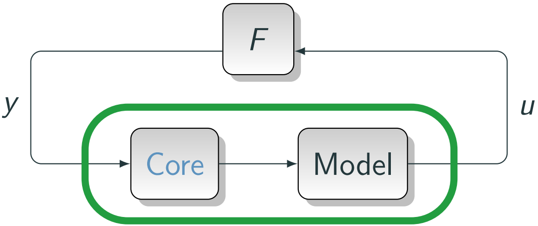
Act 2: Structure of Algorithms

Factorization of Algorithms

Regulator Equation imposes Structural requirements



Core carries all algorithm parameters

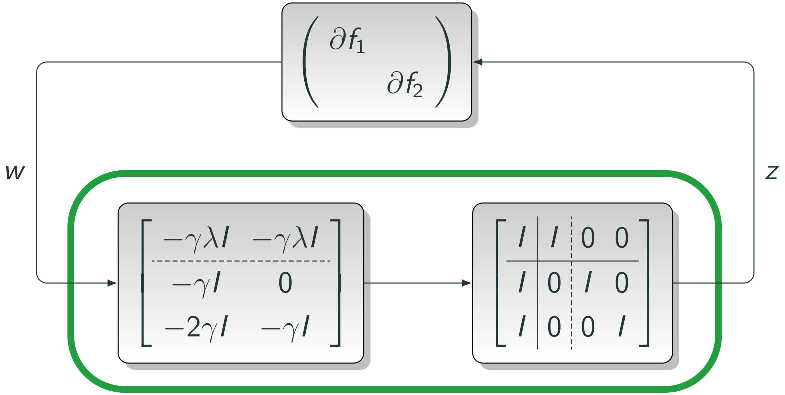
Inherited from linear Internal Model Principle (quadratics) [6] [7]

[6]Francis, Bruce A., and Walter Murray Wonham. "The internal model principle of control theory." Automatica 12.5 (1976): 457-465.

[7]Stoorvogel, Anton A., Ali Saberi, and Peddapullaiah Sannuti. "Performance with regulation constraints." Automatica 36.10 (2000): 1443-1456.

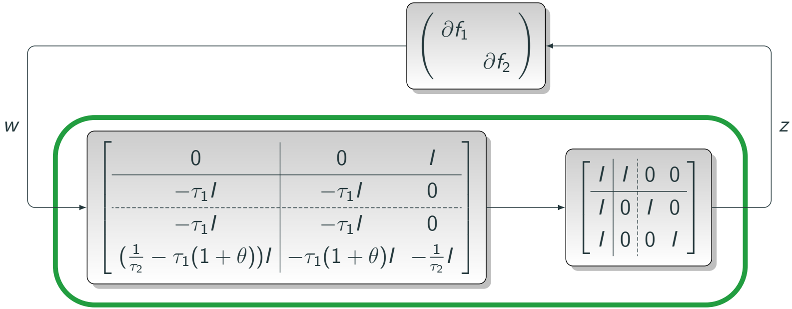
Factorization Example: Douglas-Rachford

Douglas-Rachford with parameters (λ, γ)



Factorization Example: Chambolle-Pock

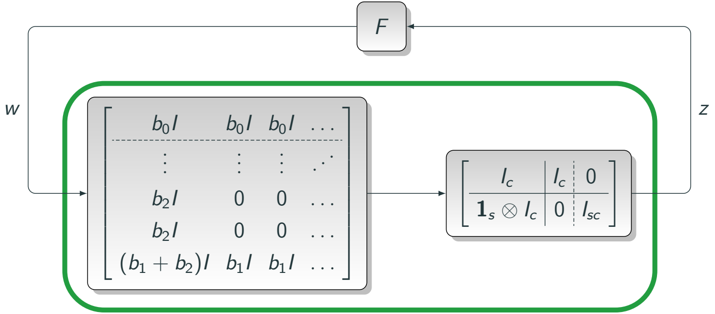
Chambolle-Pock [8] with parameters (τ_1, τ_2, θ)



[8] Chambolle, Antonin, and Pock, Thomas. "A first-order primal-dual algorithm for convex problems with applications to imaging." Journal of Mathematical Imaging and Vision 40.1 (2011): 120-145.

Factorization: Static Structure

Parameterization $(b_0, b_1, b_2) \in \mathbb{R}^3$ for common algorithms

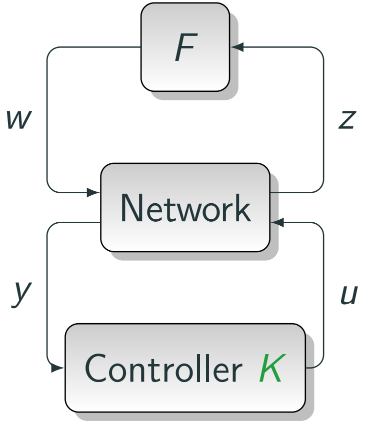


Algorithm	(b_0, b_1, b_2)
Douglas-Rachford, Davis-Yin:	$(-\gamma\lambda, -\gamma, -\gamma)$
Proximal Point, Forward-Backward:	$(-\gamma, -\gamma, 0)$

Act 3: Networked Algorithms

Algorithms over Networks

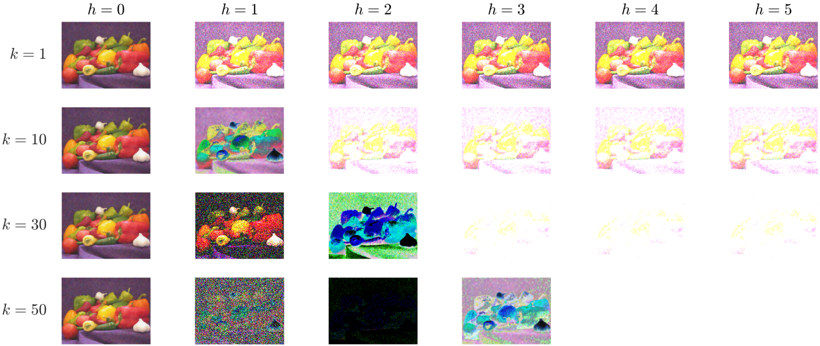
Map F may not be accessible (e.g. time-delays, cross-talk)



Interconnection of Network and Controller must be convergent

Time Delays

Time delays can destroy convergence of algorithms



Denoising: *h*-step delay on viewing image data

Convergence Conditions (Networked)

Robust Stability: Same as before (for closed-loop)

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Robust Stability: Same as before (for closed-loop)

Regulator Equation: Matrices $(\mathbf{X}_n, \mathbf{X}_c, \mathbf{U}, \mathbf{Y})$ exist with

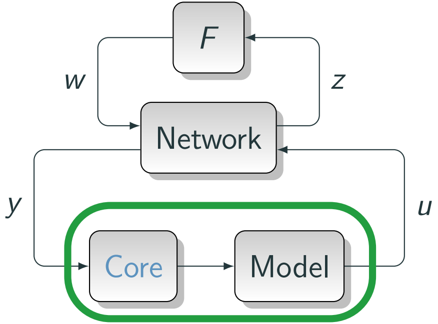
Network: $$\left(\begin{array}{c|cc:c} A & 0 & B_w N & B_u \\ \hline C_z & \mathbf{1}_s \otimes I & D_{zw} N & D_{zu} \\ \hdashline C_y & 0 & D_{yw} N & D_{yu} \end{array}\right) \left(\begin{array}{c} \mathbf{X}_n \\ \hline I \\ \hdashline \mathbf{U} \end{array}\right) = \left(\begin{array}{c} \mathbf{X}_n \\ \hline 0 \\ \hdashline \mathbf{Y} \end{array}\right),$$

Controller: $$\left(\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right) \left(\begin{array}{c} \mathbf{X}_c \\ \hline \mathbf{Y} \end{array}\right) = \left(\begin{array}{c} \mathbf{X}_c \\ \hline \mathbf{U} \end{array}\right)$$

Top equation infeasible: no convergent algorithm exists

Structural Factorization

K factors again into Core and Model



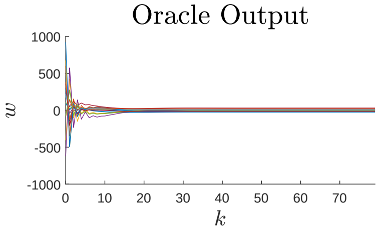
Model($\mathbf{U}, \mathbf{Y}$) depends only on network

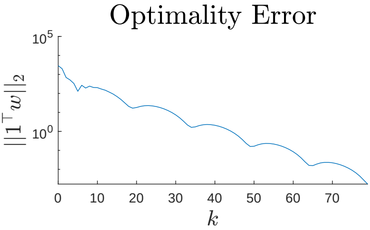
No dynamics: $u^* = \mathbf{U}d = \mathbf{1}_s \otimes \beta^*, \quad y^* = \mathbf{Y}d = w^*$

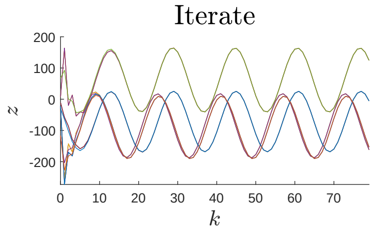
Act 4: Extensions

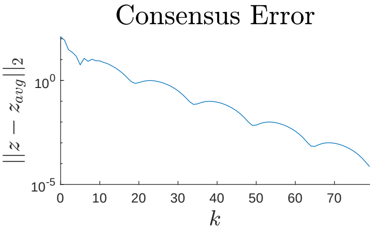
Perfect Tracking

Track the time-varying optimal solution



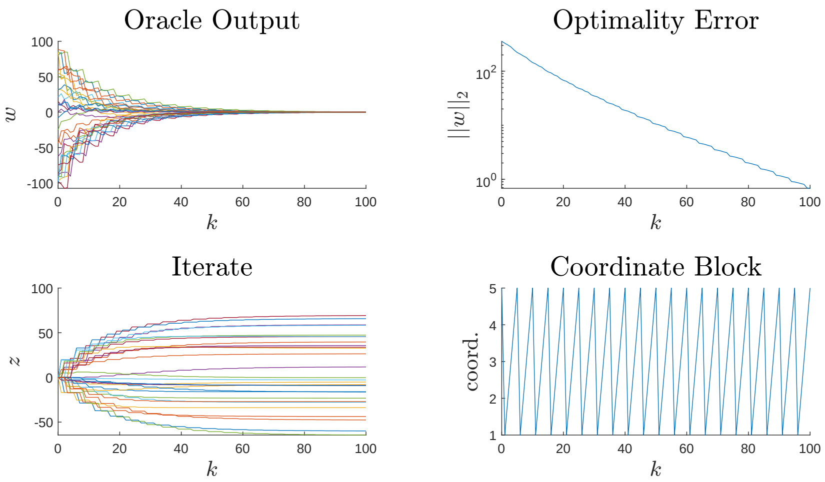






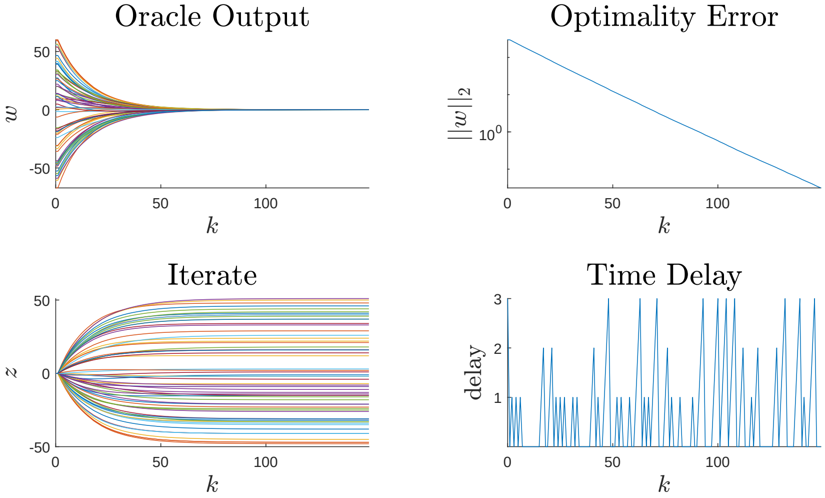
Coordinate Descent

Cyclic coordinate descent: periodic optimization



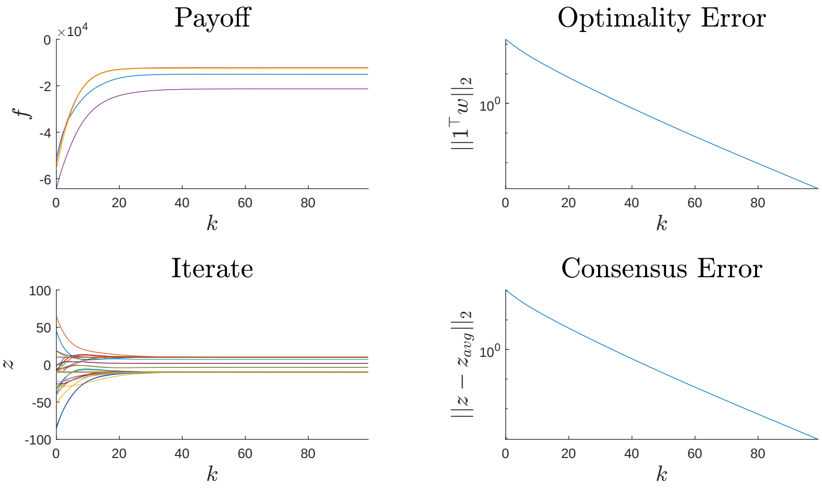
Switched Optimization Algorithms

Time-varying delay before ∂f_1



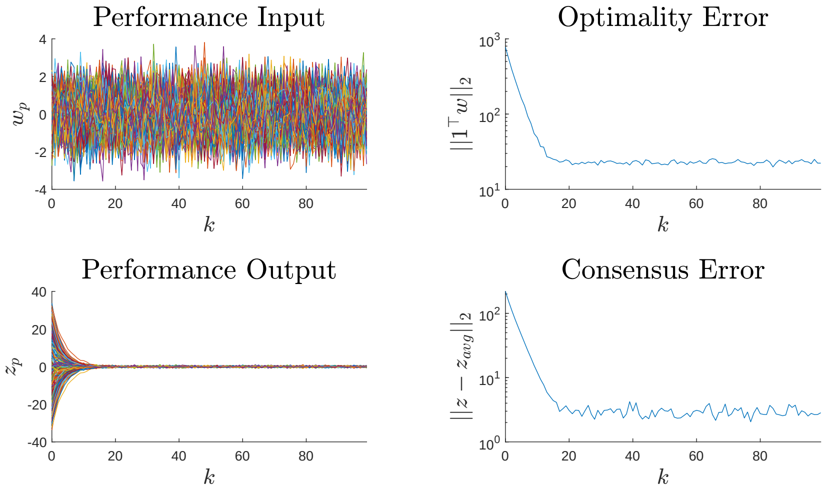
Fixed Point Equations

Games: variational Generalized Nash Equilibria



Performance

Robust/stochastic gain bounds



Act 5: Analysis and Synthesis

Exponential Convergence Rates

Analyze speed of convergence, **Synthesize** fast algorithms

Exponential convergence at rate $\rho \in (0,1)$: $\exists \gamma > 0$:

$$\|x_k - x^*\|_2 \leq \gamma \rho^k \|x_0 - x^*\|_2 \qquad \forall\ k \in \mathbb{N},\ x_0$$

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Algorithm Analysis:

Verify worst-case rate ρ over all f in class

Algorithm Synthesis:

Design algorithm with worst-case rate ρ

Filtering Framework (Analysis)

Class: $f_i - \frac{m_i}{2}\|\cdot\|_2^2$ convex, and $\frac{L_i}{2}\|\cdot\|_2^2 - f_i$ convex if $L_i < \infty$

[9] ρ-weighted generalizations of the subgradient inequality. Scherer, Carsten W., Christian Ebenbauer, and Tobias Holicki. "Optimization algorithm synthesis based on integral quadratic constraints: A tutorial." 2023 62nd IEEE Conference on Decision and Control (CDC). IEEE, 2023.

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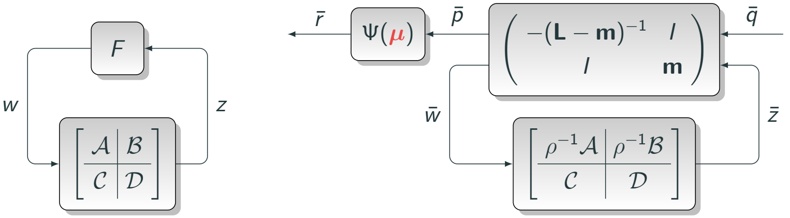


Zames-Falb filter[9] : $\Psi(\mu)$ encodes (some) valid relations

[9] ρ-weighted generalizations of the subgradient inequality. Scherer, Carsten W., Christian Ebenbauer, and Tobias Holicki. "Optimization algorithm synthesis based on integral quadratic constraints: A tutorial." 2023 62nd IEEE Conference on Decision and Control (CDC). IEEE, 2023.

Analysis Task

Linear Matrix Inequality in $(\boldsymbol{\mu}, \mathcal{X})$ for fixed ρ:

$$\mathcal{X} \succ 0, \;\; \boldsymbol{\mu}_0^i > 0, \quad \boldsymbol{\mu}_t^i \geq 0,$$

$$[*]^\top \begin{bmatrix} \mathcal{X} & 0 \\ 0 & -\mathcal{X} \end{bmatrix} \begin{bmatrix} \hat{\mathcal{A}}(\boldsymbol{\mu}) & \hat{\mathcal{B}}(\boldsymbol{\mu}) \\ I & 0 \end{bmatrix} + [*]^\top \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathcal{C}}(\boldsymbol{\mu}) & \hat{\mathcal{D}}(\boldsymbol{\mu}) \\ 0 & I \end{bmatrix} \prec 0$$

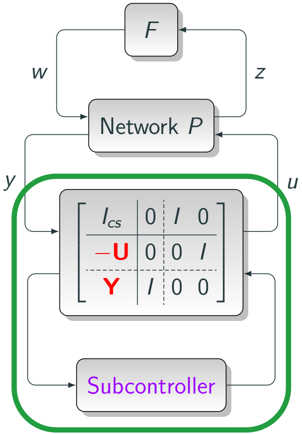
Algorithm ρ-convergent *and well-posed* if (sufficient)

1. Regulator Equation satisfied
2. LMI is feasible (strict negative realness)
3. $\mathcal{D}$ is block-lower-triangular

Full-Order Internal Models and Synthesis

Add a model in the middle

Regulator Equation *always* satisfied

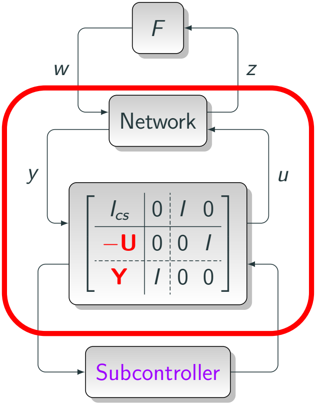


Full-Order Internal Models and Synthesis (cont.)

Accumulate Network and Model

Convex reduced-order design of Subcontroller

Allows for performance
(e.g. convergence rates, tracking)



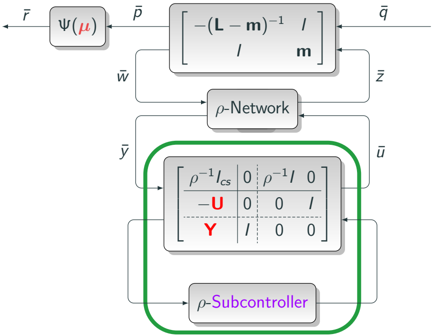
Alternating Synthesis

Alternating convex problem

Bisection in ρ: inner steps

Start with $\Psi(\mu) = I_{cs}$

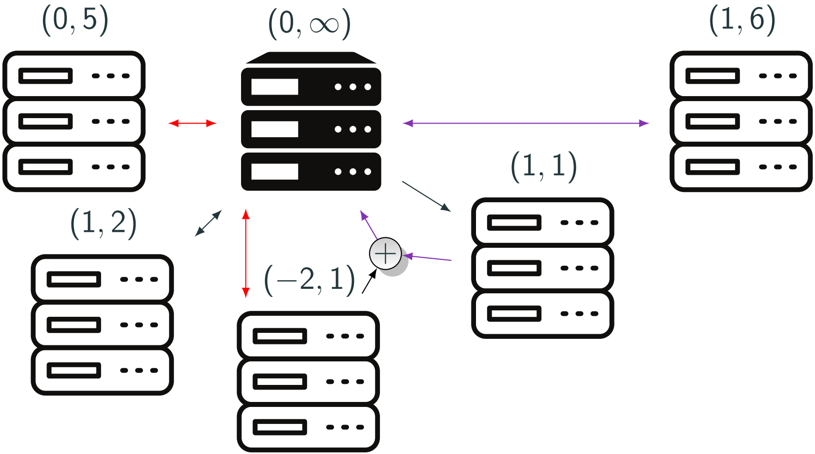
Alternating search between μ, Subcontroller



LMI solvable $\bar{q} \to \bar{r}$

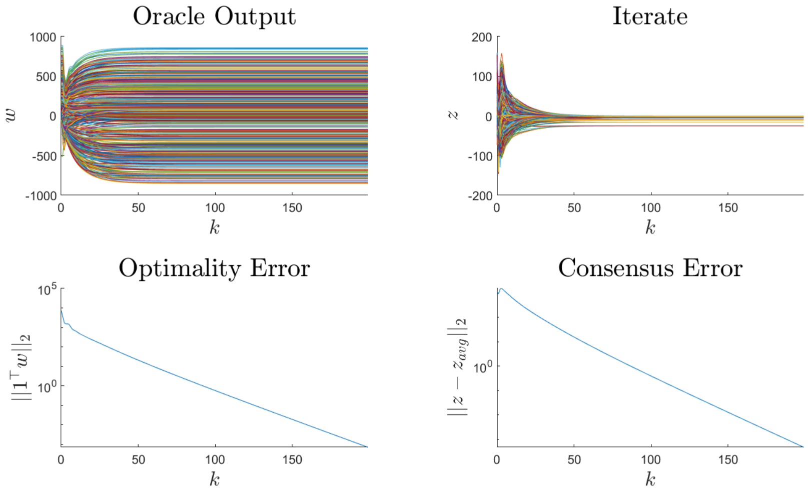
Server Example: With Network Dynamics

Sparse $\beta \in \mathbb{R}^{200}$: $f_{1:5}$ quadratics, $f_6 = \chi_{\|\cdot\|_1 \le 100}$, $\|\beta^*\|_0 = 13$



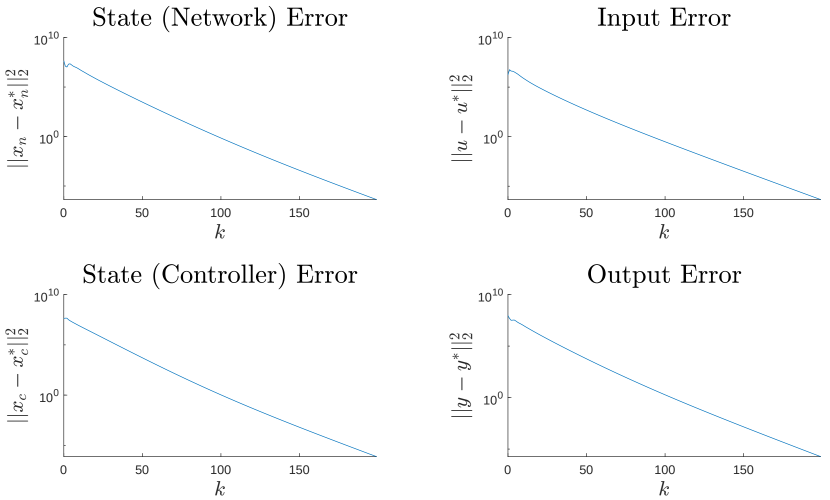
Server Example: Convergence

$\rho \leq 0.95$ algorithm: order(Network) $= 9c$, order(K) $= 15c$,



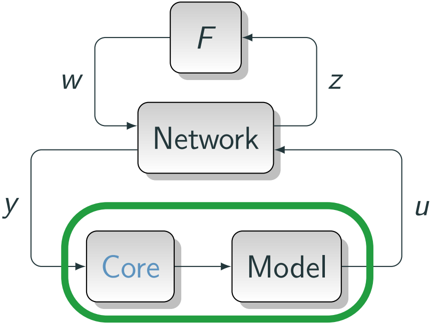
Server Example: Tracking

Quantities (x_n, x_c, y, u) all track optimal solution



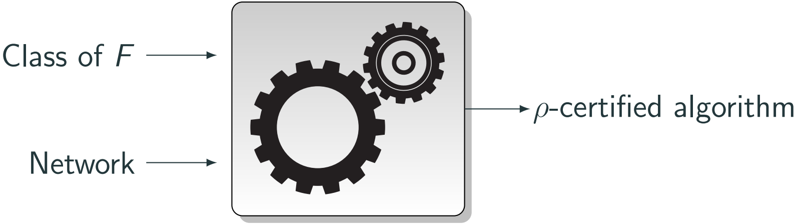
Coda

Conclusion

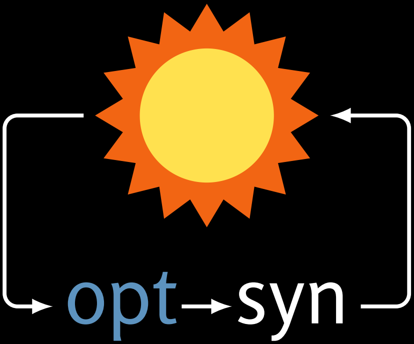


Convergence

Structure



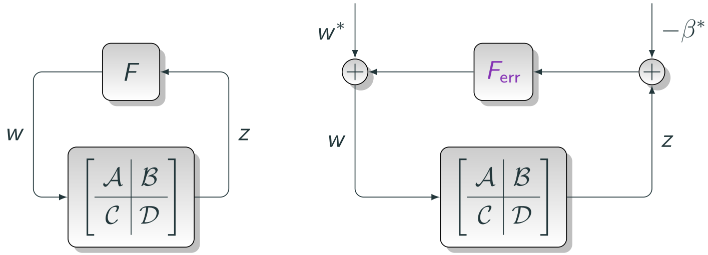
Optimize(d) using Structure!



Bonus Slides

Generalized Plant Perspective

Error nonlinearity centered at $0 \in F_{\text{err}}(0)$

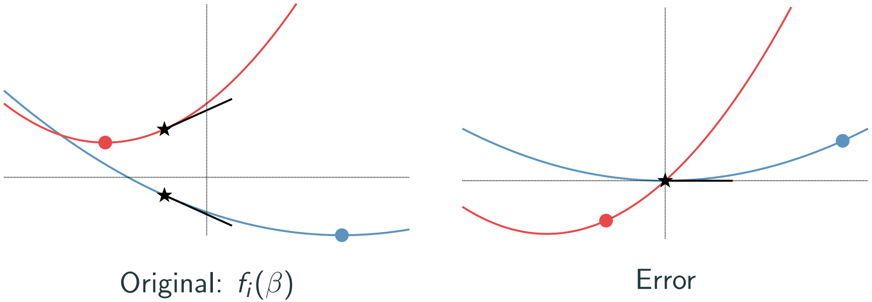


When F affine, this is a linear system disturbed by (β^*, w^*)

Details : $$f_{\text{err},i}(e_i) = f_i(e_i + \beta_i^*) - {w^*}^i(e_i + \beta_i^*) - f_i(\beta^*)$$

Error System

Convergence requires $\beta^* = 0$



Error Model

Consensus Parameterization

Robust control theory requires convergence to 0, not to β^*

Parameterize subgradient using $\hat{w}^* \in \mathbb{R}^{(s-1)c}$ as $w^* = N\hat{w}^*$

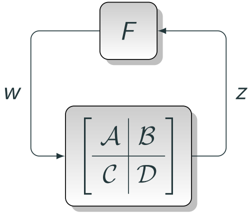
$$N = \begin{pmatrix} I_{s-1} \\ \mathbf{1}_s^\top \end{pmatrix} \otimes I_c, \qquad \text{at } s = 3\text{: } N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix} \otimes I_c$$

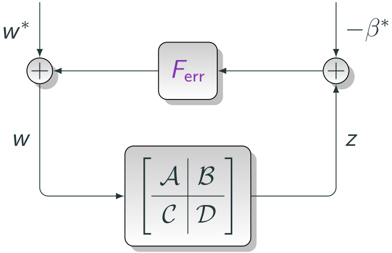
Drive errors to 0: $\quad \tilde{w} := w - N\hat{w}^*, \quad \tilde{z} := z - \mathbf{1}_s \otimes \beta^*$

Error System

Zero-center the nonlinearity F using error map $\tilde{F}$

Error map[10] $\tilde{F} \in \mathcal{O}^0_{m,L}$, $\qquad \tilde{F}(\tilde{z}) := F(\tilde{z} + \mathbf{1}_s \otimes \beta^*) - N\hat{w}^*$





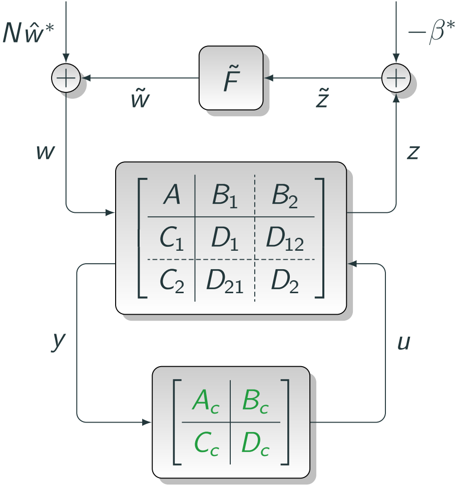
[10] Wu, Wuwei, et al. "Frequency-Domain Synthesis of Implicit Algorithms." 2025 American Control Conference (ACC). IEEE, 2025.

Error System: Networked

Constant disturbances
$(\beta^*, \hat{w}^*)$ perturb $P \star K$

Want to design K such that
$(\tilde{w}, \tilde{z}) \to 0 \quad \forall \tilde{F} \in \mathcal{O}^0_{m,L}$

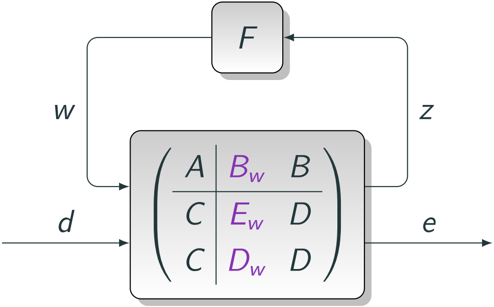
How?



Disturbance Rejection through Maps

F is zero-centered ($0 \in F(0)$)

Disturbance rejection: want $e \to 0$ for all (x_0, d, F)



Regulation through Maps

If linear map $F(z) = \Delta z$ is applied (quadratic objective), then

Stability: $A + B_w\Delta(I - E_w\Delta)^{-1}C$ is Schur

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Regulator Equation: **Same** as before

$$\exists \mathbf{X} : \qquad \begin{pmatrix} A - I & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Cancellation of Δ contribution by regulator equation

$$\begin{pmatrix} x_{k+1} \\ \hline e_k \end{pmatrix} = \left[\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) + \text{Something}(\Delta) \left(\begin{array}{c|c} C & D \end{array} \right) \right] \begin{pmatrix} x_k \\ \hline d \end{pmatrix}$$

Extends to zero-centered nonlinearities F if well-posed

Convex Functions: Subgradient Inequality

If f is convex, subgradient inequality implies $\forall w \in \partial f(z)$

$$f(z) - f(z') \leq w^\top (z - z')$$

and if $z' = 0,\ f(z') = 0$: $\qquad 0 \leq f(z) \leq w^\top z$

[11] Van der Schaft, A. (2000). L2-gain and passivity techniques in nonlinear control. Berlin, Heidelberg: Springer Berlin Heidelberg.

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Any sequence $\{(w_k, z_k)\}_{k=-\infty}^{\infty}$ with $w_k \in \partial f(z_k)$ satisfies [11]

$0 \leq f(z_k) \leq w_k^\top z_k \ \forall k,$ implying *passivity* $0 \leq \sum_{k=0}^{\infty} w_k^\top z_k$

[11] Van der Schaft, A. (2000). L2-gain and passivity techniques in nonlinear control. Berlin, Heidelberg: Springer Berlin Heidelberg.

Convex Functions: Zames-Falb

We have static and dynamic relations:

$$\begin{array}{rr} \text{Static} & 0 \leq \sum_{k=0}^{\infty} w_k^\top z_k \\ \text{Dynamic} & \forall k:\ 0 \leq \sum_{k=0}^{\infty} w_k^\top (z_k - z_{k-t}) \end{array}$$

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Static	$0 \le \sum_{k=0}^{\infty} w_k^\top z_k$
Dynamic	$\forall k: \; 0 \le \sum_{k=0}^{\infty} w_k^\top (z_k - z_{k-t})$

For *any* nonnegative coefficients $\{\mu_t\}_{t=0}^{\infty}$, we add them up:

$$0 \le \sum_{k=0}^{\infty} \left(\mu_0 w_k^\top z_k\right) + \left(\sum_{t=1}^{\infty} \mu_t w_k^\top (z_k - z_{k-t})\right)$$

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This forms a *Zames-Falb* filter $\Psi(\mu)$:

$$r_k := \left(\sum_{t=0}^{\infty} \mu_t\right) z_k - \sum_{t=1}^{\infty} \mu_t z_{k-t} \qquad \sum_{k=0}^{\infty} w_k^\top r_k \geq 0$$

Strongly Convex Functions

Strong convexity with parameter $m > 0$:

$$f(z') \geq f(z) + w^\top (z' - z) + \frac{m}{2}\|z' - z\|_2^2, \qquad f - \frac{m}{2}\|\cdot\|_2^2 \text{ is convex}$$

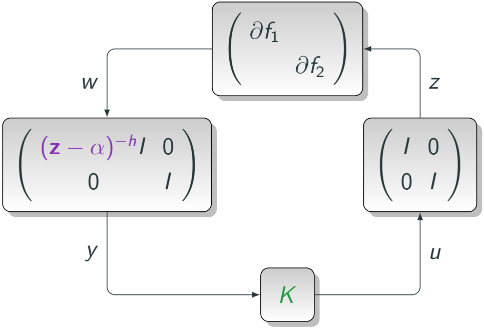
Same Zames-Falb filter $\Psi(\mu)$: [12]

$$r_k := \left(\sum_{t=0}^{\infty} \mu_t\right) z_k - \sum_{t=1}^{\infty} \mu_t z_{k-t} \qquad \sum_{k=0}^{\infty} (w_k - m z_k)^\top r_k \geq 0$$

[12] Scherer, C. W. (2023). Robust exponential stability and invariance guarantees with general dynamic O'Shea-Zames-Falb multipliers. IFAC-PapersOnLine, 56(2), 5799-5804.

Channel Memory: Network Example

Network P^{α}: pole of order h introduced



Regulator Equation unsolvable at $\alpha = 1$ (integrator/ramp)

Channel Memory: Regulation

Controllers that satisfies Regulator Equation if $\alpha \neq 1$

$$K_h^\alpha[b] : \qquad \left(\begin{array}{c|cc} I & (1-\alpha)^h b_0 I & b_0 I \\ \hline I & (1-\alpha)^h b_2 I & 0 \\ I & (1-\alpha)^h (b_1 + b_2) I & b_1 I \end{array}\right)$$

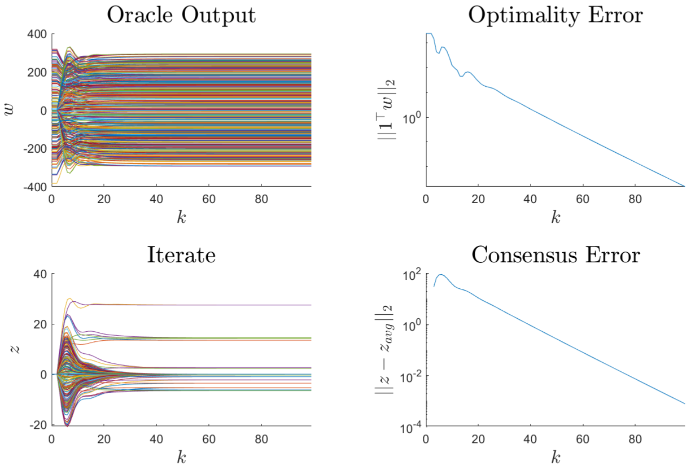
Douglas Rachford[13]: $(b_0, b_1, b_2) = (-\gamma\lambda, -\lambda, -\lambda)$ with $\alpha = 0$

Still need to check Robust Stability

[13] Douglas, J., & Rachford, H. H. (1956). On the numerical solution of heat conduction problems in two and three space variables. Transactions of the American Mathematical Society, 82(2), 421-439.

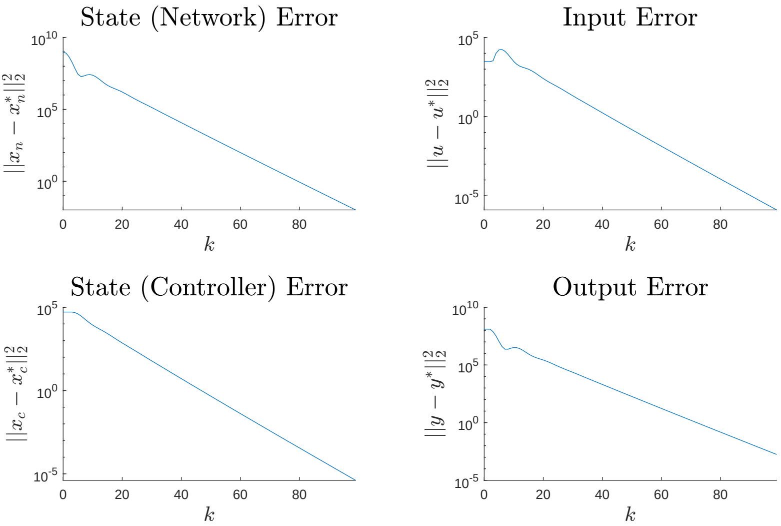
Bonus: Channel Memory

Channel Memory: Convergence



LASSO solved using $K_3^{0.5}[-0.04, -0.2, -0.1]$

Channel Memory: Tracking



Asymptotic tracking of other signals